Period map for non-compact holomorphically symplectic manifolds

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Abstract: We study the deformations of a holomorphic symplectic manifold M, not necessarily compact, over a formal ring. We show (under some additional, but mild, assumptions on M) that the coarse deformation space exists and is smooth, finite-dimensional and naturally embedded into $H^2(M)$. For a holomorphic symplectic manifold M which satisfies $H^1(\mathcal{O}_M) = H^2(\mathcal{O}_M) = 0$, the coarse moduli of formal deformations is isomorphic to $\operatorname{Spec} \mathbb{C}[[t_1, ..., t_n]]$, where t_1, ... t_n are coordinates in $H^2(M)$.

Contents

1   Introduction

Deformations and moduli of compact Kähler manifolds are a well studied subject, dating back to Kodaira-Spencer [KS]. The moduli of non-compact manifolds are rarely mentioned, mostly because they are much harder to define and study.

The work on the moduli of compact holomorphically symplectic manifolds and Calabi-Yau manifolds is still far from the conclusion; the local case is due to F.Bogomolov, A.Beauville, G.Tian, A.Todorov, P.Deligne and Z.Ran ([Bo], [Bea], [T], [To], [R]). Extreme importance of this subject is highlighted by thousands of papers on Mirror Symmetry, which appeared since then.

In the non-compact case, some work in this direction was done by M.Kontsevich and S.Barannikov ([BK]) and others, but, for the most part, this territory is still uncharted. However, there are many examples that suggest that at least for some non-compact holomorphically symplectic manifolds a good local deformation theory does exist. In particular, in the well-studied case of smooth crepant resolutions of symplectic quotient singularities in dim 2 (the so-called Du Val points), a likely candidate for the universal local deformation is provided by the simultaneous resolution of Brieskorn [Br].

In this paper we extend to the non-compact case the algebraic construction of the local deformation space of a Calabi-Yau manifold M, due to Z.Ran. Unfortunately, our results are valid only when the manifold M is holomorphically symplectic.

Our approach is essentially the same as the original approach of Bogomolov. It is based on the so-called **period map**. Instead of deformations of a holomorphically symplectic manifold M, one considers deformations of the pair $\langle M, \Omega \rangle$, where Ω denotes the holomorphic symplectic form, that is, a nowhere degenerate closed $(2, 0)$-differential form. Given a local deformation $\pi : \ \widetilde{M} \longrightarrow S$ of $\langle M, \Omega \rangle$ with a simply connected base, the cohomology of the individual fibers of π are identified by the Gauss-Manin connection. Taking the cohomology class of the holomorphic symplectic form of each fiber, one obtains a map $\mathsf{Per} : \ S \longrightarrow H^2(M)$. Bogomolov and Beauville have shown that for M compact, the map Per induces a holomorphic immersion of the coarse marked moduli space $\mathcal{M}$ of $\langle M, \Omega \rangle$ into $H^2(M)$. The image of Per belongs to a certain quadric $\mathcal{C} \subset H^2(M)$, cut by the so-called **Bogomolov-Beauville form**. Moreover, the period map $\mathsf{Per} : \ \mathcal{M} \longrightarrow \mathcal{C}$ is locally an isomorphism.

Bogomolov extended these results to Calabi-Yau manifolds in an un-

published I.H.E.S preprint (1982). In 1987, Tian and Todorov published a different proof of Bogomolov's theorem. Their proof of Bogomolov-Tian-Todorov theorem was based on Hodge theory. An algebraic version of their arguments was proposed by Z.Ran ([R]). Ran's argument uses the degeneration of the E_2-term of the Dolbeault spectral sequence (proven in algebraic case by Deligne and Illusie, [DI]). However, this spectral sequence is not degenerate in non-compact case, hence this argument does not work for open Calabi-Yau manifolds.

We found that a version of Z.Ran's argument is valid for holomorphic symplectic manifolds (under some additional, quite weak, assumptions). One can explain this heuristically as follows. For a complex manifold M, deformations are classified by $H^1(\mathcal{T}(M))$, where $\mathcal{T}(M)$ is the tangent sheaf. When M is Calabi-Yau, $\mathcal{T}(M)$ is isomorphic to $\Omega^{n-1}(M)$, where $n = \dim_{\mathbb{C}} M$. To show that the deformations of M have no obstructions, we would need to prove that the E_2-term of the Dolbeault spectral sequence degenerates in $H^1(\Omega^{n-1}(M))$. This is very far from truth in the non-compact case. However, if M is holomorphic symplectic, we have $\mathcal{T}(M) \cong \Omega^1(M)$, and instead of $H^1(M, \Omega^{n-1}(M))$ we have to consider $H^1(M, \Omega^1(M))$. The only differential in the spectral sequence that maps into $H^1(M, \Omega^1(M))$ starts at $H^1(M, \mathcal{O}_M)$. If we assume for simplicity that $H^i(M, \mathcal{O}_M) = 0$ for $i \geq 1$, then this differential vanishes tautologically. The differentials that *start* at $H^1(M, \Omega^1(M))$ can still be non-trivial, but it turns out that they become irrelevant if instead of deformations of M one considers deformations of the pair $\langle M, \Omega \rangle$.

Thus in the case $H^i(M, \mathcal{O}_M) = 0$, $i \geq 1$ we obtain the following result.

Theorem 1.1. *Let M be a holomorphic symplectic manifold such that for every $i \geq 1$ we have $H^i(\mathcal{O}_M) = 0$. Then there exists a coarse moduli space of formal deformations*

$$\pi : \ \widetilde{M} \longrightarrow Spl(M, \Omega)$$

of pairs $\langle M, \Omega \rangle$. Moreoved, the period map Per $: \ Spl(M, \Omega) \longrightarrow H^2(M)$ *gives an isomorphism of $Spl(M, \Omega)$ and the formal completion of $H^2(M)$ in $[\Omega] \in H^2(M)$.* $\qquad \square$

There is a more general version of this result (Theorem 3.6), which works for a larger class of non-compact holomorphically symplectic manifolds. Here we also obtain a coarse moduli space $Spl(M, \Omega)$, which is smooth and finite-dimensional, but it is no longer isomorphic to $H^2(M)$. However, the period map remains an immersion.

We will now give a semi-rigorous sketch of the proof of this result. First, consider an easy but important example of affine holomorphic symplectic manifold M. We have $H^1(\mathcal{T}(M)) = 0$, hence any first-order complex deformation of M is trifial. Using induction, it is easy to show that any formal complex deformation of M is also trivial, that is, for any formal deformation $\pi: \widetilde{M} \longrightarrow S$ of M, we have $\widetilde{M} \cong S \times M$. However, a symplectic deformation needs not to be trivial, because we may have non-trivial variations of holomorphic symplectic structure. A formal deformation of the pair $\langle M, \Omega \rangle$ is determined by the deformation of a closed $(2,0)$-form Ω.

By Grothendieck's theorem, the topological cohomology of M is isomorphic to the hypercohomology of the algebraic de Rham complex

$$(1.1) \qquad 0 \longrightarrow \mathcal{O}_M \xrightarrow{\partial} \Omega^1 M \xrightarrow{\partial} \Omega^2 M \xrightarrow{\partial} \ldots$$

Since M is affine, $H^i(\Omega^j(M)) = 0$ for $i > 0$. Therefore, $H^2(M)$ is isomorphic to the second cohomology of the complex (1.1). A first order infinitesimal automorphism of M is given by the section of TM, which is isomorphic to $\Omega^1(M)$. A (co-)vector field $\gamma \in \Omega^1(M) \cong \mathcal{T}(M)$ acts on $\Omega^i M$ by the Lie derivative. Therefore, γ acts on a closed form $\Omega \in \Omega^2(M)$ by adding $d\gamma$, and the first-order deformations of $\langle M, \Omega \rangle$ are classified by closed 2-forms up to exact 2-forms. Using induction, it is easy to check that this is true in any order. Therefore, the period map gives an isomorphism of the coarse moduli of $\langle M, \Omega \rangle$ and the formal neighbourhood of the class $[\Omega]$ in $H^2(M)$.

The general proof is deduced from the affine version as follows.

Let S be a spectrum of an Artin ring over $\mathbb{C}$. We consider the deformations of $\langle M, \Omega \rangle$ over S as a stack of groupoids over M. We introduce another stack, called **Kodaira-Spencer stack**, which – whenever it is defined – classifies the maps from M to $H^2(M)$. Using the affine version of the main theorem, we show that these stacks are equivalent over affine open subsets of M. This immediately implies that these stacks are equivalent over the whole M, which concludes the study of the formal deformations of $\langle M, \Omega \rangle$ and shows that these deformations are classified by the maps form the base of deformation to $H^2(M)$.

The above account is greatly simplified. To make it at least approximately workable, we have to do big technical adjustments. The problems are twofold: first of all, the Kodaira-Spencer groupoid is not defined in a general situation – we have to go step-by-step through what we call *elementary* base extensions to define it properly. Secondly, the Kodaira-Spencer groupoid classifies not the maps from the base S to $H^2(M)$, but rather a certain maps of complexes of sheaves, which are reduced to the maps from S to $H^2(M)$ when $H^1(\mathcal{O}_M) = H^2(\mathcal{O}_M) = 0$.

Now we give a more precise version of the definition of the Kodaira-Spencer stack. Let S be an Artin scheme over $\mathbb{C}$, and $S_0 \subset S$ a closed subscheme defined by an ideal $I \subset \mathcal{O}_S$, such that $I^2 = 0$. Assume that the ideal I is sufficiently small (see Definition 4.3 for the precise condition on I). Fix a deformation $\pi_0 : \widetilde{M}_{S_0} \longrightarrow S_0$ of $\langle M, \Omega \rangle$. Consider the set $\mathsf{Def}(\widetilde{M}_{S_0}, S)$ of all deformations $\widetilde{M}_S$ of $\langle M, \Omega \rangle$ over S, equipped with an isomorphism

$$\widetilde{M}_S \times_S S_0 \longrightarrow \widetilde{M}_{S_0}.$$

Clearly, $\mathsf{Def}(\widetilde{M}_{S_0}, S)$ is a stack of groupoids over M. Consider a truncated relative de Rham complex $F^1 \Omega^*(\widetilde{M}_{S_0}/S_0)$,

$$\Omega^1(\widetilde{M}_{S_0}/S_0) \xrightarrow{\ \partial\ } \Omega^2(\widetilde{M}_{S_0}/S_0) \xrightarrow{\ \partial\ } \ldots$$

A holomorphic symplectic structure on the deformation $\widetilde{M}_{S_0}$ defines a morphism of complexes $\pi_0^* \mathcal{O}_{S_0}[2] \longrightarrow F^1 \Omega^*(\widetilde{M}_{S_0}/S_0)$. Taking its derivative along the Gauss-Manin connection on S_0, we obtain the so-called Kodaira-Spencer map of the deformation $\widetilde{M}_{S_0}$

$$\pi_0^* T S_0[2] \xrightarrow{\ \theta_0\ } F^1 \Omega^*(\widetilde{M}_{S_0}/S_0).$$

The Kodaira-Spencer stack $KS(\widetilde{M}_{S_0}, S)$ is defined as follows. The *objects* of $KS(\widetilde{M}_{S_0}, S)$ are all morphisms of complexes

$$\pi_0^* T S \otimes_{\mathcal{O}_S} \mathcal{O}_{S_0}[2] \xrightarrow{\ \theta\ } F^1 \Omega^*(\widetilde{M}_{S_0}/S_0)$$

such that the restriction of θ to $T S_0$ is equal to θ_0. The *morphisms* between any two such θ_1, θ_2 are chain homotopies between them – that is, maps

$$\pi_0^* T S \otimes_{\mathcal{O}_S} \mathcal{O}_{S_0}[1] \xrightarrow{\ \gamma\ } F^1 \Omega^*(\widetilde{M}_{S_0}/S_0)$$

satisfying $d\gamma = \theta_1 - \theta_2$. This obviously defines a groupoid. The definition of the period map from $\mathsf{Def}(\widetilde{M}_{S_0}, S)$ to $KS(\widetilde{M}_{S_0}, S)$ is straightforward; using the deformation theory for affine M, we show that this is an isomorphism. In other words, there are exactly as many ways to extend the deformation from S_0 to S as there are ways to extend the corresponding morphism to $H^2(M)$. Using induction, this leads immediately to the classification result for the deformations stated above.

The paper is organized as follows. We start with a semi-heuristic analytic proof of the main theorem obtained by the use of the Cartan-Maurer

equation, as suggested by Kontsevich and Barannikov. This proof cannot be made precise in the non-compact situation, but we have decided to include it anyway, since it exhibits nicely the general idea behind the rigorous proof. Then we turn to purely algebaric methods. Section 3 contains the necessary definitions and the precise statement of our result in the strongest form. In Section 4, we introduce the approriate symplectic version of the Kodaira-Spencer class and define elementary extensions. In Section 5, we rework our construction in the language of stacks and prove the main theorem. Finally, Section 6 is a short postface which explains how our results are related to the known facts (in particular, to the results of Z. Ran).

2 Deformations in mixed formal complex-analytic category

One of the way to study deformations us through the DG-algebra approach suggested by Kontsevich and Barannikov ([Ba], [BK]). The main advantage of this method is that it foregoes the tedious step-by-step constructions of Grothendieck's local deformations theory, and gives the results in the form of an explicit power series. Unfortunately, it is not quite as useful in non-compact case, when we have no means to check that these series converge.

The deformations one obtains from the Dolbeault complex lie a weird mixed formal-complex-analytic category. To obtain something definite, one needs some kind of integrability-type conditions. Kontsevich and Barannikov work with compact manifolds, so that in their situation this is not a problem – integrability can be obtained directly from functional analysis. However, since we study open manifolds, the functional analysis does not help, and we stay in the mostly useless mixed category.

Nevertheless, Kontsevich's approach (which goes back to Kodaira) is beautiful and quite useful as a heuristic tool. Therefore we decided to express some of our results in this language before going to the rigourous step-by-step proof.

Since the DG-algebra approach is used only for heuristics, this section will be quite sketchy; a cursory knowledge of [BK] is required.

We start with a review of the deformation theory as it is given in [Ba] and [BK]. Let M be a complex manifold, $M_{\mathbb{R}}$ the underlying real analytic manifold and $M_{\mathbb{R}}[[t]] := M_{\mathbb{R}} \times \operatorname{Spec}(\mathbb{C}[[t]])$ the "mixed formal-real analytic" manifold obtained as a product of $M_{\mathbb{R}}$ and the formal disk $\Delta := \operatorname{Spec}(\mathbb{C}[[t]])$. Using DG-algebras, one may classify the complex deformations of M over Δ, that is, the complex structures J on $M_{\mathbb{R}}[[t]]$ such that the zero fiber of

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$(M_\mathbb{R}[[t]], J)$ is isomorphic to M. It is well known that such deformations are classified by the solutions of the Maurer-Cartan equation

(2.1) $$\bar\partial \gamma(t) = -\frac{1}{2}[\gamma(t), \gamma(t)],$$

where $\gamma(t) \in \Lambda^{0,1}(TM)[[t]]$ is a $C[[t]]$-valued $(0,1)$-form with coefficients in holomorphic vector fields, and

$$[\cdot, \cdot] : \ \Lambda^{0,1}(TM)[[t]] \times \Lambda^{0,1}(TM)[[t]] \longrightarrow \Lambda^{0,2}(TM)[[t]]$$

is the Schouten bracket.

We explain in a few words how the complex structures are classified by the solutions of Maurer-Cartan.

Consider the sheaf $A^{\bullet,\bullet} = \Lambda^{\bullet,\bullet}(M)$ as an abstract sheaf of algebras. A complex structure on M is defined by an identification between $\Lambda^1(M)$ and $A^{0,1}$. Since $\Lambda^1(M)$ classifies the derivations of the sheaf of smooth functions, to give an identification $\Lambda^1(M) \cong A^{0,1}$ is the same as to give a derivation $\bar\partial : \ C^\infty(M) \longrightarrow A^{0,1}$. The difference between two such operators is given by $\gamma \in \Lambda^{0,1}(TM)$. An integrability condition is written as $(\bar\partial + \gamma)^2 = 0$, which is rewritten as the Maurer-Cartan equation. Deformations are equivalent if the corresponding operators are exchanged by an automorphism of $A^{\bullet,\bullet}$.

We are going to write a similar interpretation for holomorphic symplectic deformations.

Fix a holomorphic symplectic form $\Omega \in A^{2,0}$. A holomorphic symplectic deformation is defined by an operator

$$\bar\partial + \gamma : \ C^\infty(M) \longrightarrow A^{0,1}, \quad \gamma(\Omega) = 0$$

such that $(\bar\partial + \gamma)^2$, or, equivalently, γ satisfies the Maurer-Cartan equation (2.1). Deformations are equivalent if the corresponding operators are exchanged by an automorphism of $A^{\bullet,\bullet}$ preserving Ω.

In this spirit, Barannikov [Ba] describes the deformations of Calabi-Yau manifolds, with Ω a nowhere degenerate section of the sheaf of top-degree $(p, 0)$-forms.

For the holomorphic symplectic manifolds, the deformations are described in terms the sheaf $\mathcal{H}am$ of holomorphic Hamiltonian vector fields. First of all, automorphisms of $A^{\bullet,\bullet}$ preserving Ω correspond to the Hamiltonian vector fields $\beta \in \mathcal{H}am(M)$. Secondly, the condition $\gamma\Omega = 0$ means that $\gamma \in \Lambda^{0,1}(\mathcal{H}am(M))$.

We obtain that the first-order deformations of M are described by the cohomology of the sheaf of Hamiltonian vector fields. To prove that the deformations of M are unobstructed (that is, every one-parametric first-order deformation is extended to a deformation over a formal disk), we need to show the following. Let $\gamma_1 \in \Lambda^{0,1}(\mathcal{H}am(M))$ be a $\bar{\partial}$-closed $(0,1)$-form with coefficients in $\mathcal{H}am$. Then there exists a series $\gamma_2, \gamma_3, \cdots \in \Lambda^{0,1}(\mathcal{H}am(M))$ such that

$$(\bar{\partial} + t\gamma_1 + t^2\gamma_2 + \dots)^2 = 0,$$

that is, the formal series $\gamma_1 + t\gamma_2 + t^2\gamma_3 + \dots$ satisfies the Maurer-Cartan equation (2.1). In this setting, Maurer-Cartan can be written as follows:

$$(2.2) \qquad \bar{\partial}\gamma_{n+1} = -\frac{1}{2}\sum_{i+j=n}[\gamma_i,\gamma_j], \quad n = 2,3,\dots$$

Consider the following commutative diagram

$$(2.3) \qquad
\begin{array}{ccccc}
\xrightarrow{\ \bar{\partial}\ } & \Lambda^{2,1} & \xrightarrow{\ \bar{\partial}\ } & \Lambda^{2,2} & \xrightarrow{\ \bar{\partial}\ } \\
 & \big\uparrow{\scriptstyle\partial} & & \big\uparrow{\scriptstyle\partial} & \\
\xrightarrow{\ \bar{\partial}\ } & \Lambda^{1,1} & \xrightarrow{\ \bar{\partial}\ } & \Lambda^{1,2} & \xrightarrow{\ \bar{\partial}\ } \\
 & \big\uparrow{\scriptstyle\partial} & & \big\uparrow{\scriptstyle\partial} & \\
\xrightarrow{\ \bar{\partial}\ } & \Lambda^{0,1} & \xrightarrow{\ \bar{\partial}\ } & \Lambda^{0,2} & \xrightarrow{\ \bar{\partial}\ }
\end{array}$$

Identifying TM and $\Lambda^{1,0}$, we can realize the polyvector fields as differential (p,q)-forms. In particular, the solutions γ_i of (2.2) are considered now as sections of $\Lambda^{1,1}$. The Hamiltonian condition is written as $\partial\gamma_i = 0$, where $\partial : \Lambda^{1,1} \longrightarrow \Lambda^{2,1}$ is an operator in (2.3). Consider the "symplectic Hodge operator"

$$\Lambda : \Lambda^{p,q} \longrightarrow \Lambda^{p-2,q}$$

which is adjoint to the multiplication by the holomorphic symplectic form Ω via the non-degenerate product defined by Ω. The Schouten bracket is written in terms of Λ and ∂ as follows.

Lemma 2.1. **(Tian-Todorov lemma for holomorphic symplectic manifolds.)** *Let* $\gamma, \gamma' \in \Lambda^{1,1}$, *and let* $[\gamma, \gamma'] \in \Lambda^{2,1}$ *denote the Schouten*

*bracket (we identify TM and $\Lambda^{1,0}$ using the holomorphic symplectic form).
Then*

$$[\gamma, \gamma'] = \partial\Lambda(\gamma \wedge \gamma') - \Lambda(\partial\gamma \wedge \gamma') - \Lambda(\gamma \wedge \partial\gamma').$$

Proof. This is a symplectic version of the standard Tian-Todorov lemma, and it is proven in exactly the same fashion as the usual one ([To]). $\qquad\square$

The main result of this Section is the following theorem, which states that the deformations of holomorphic symplectic manifold are unobstructed, given that $H^2(\mathcal{O}_M) = 0$. This assumptions is not essential, but it makes the statements much simpler.

Theorem 2.2. *Let M be a complex holomorphic symplectic manifold, and let $\gamma_1 \in \Lambda^{0,1}(\mathcal{H}am)$ be a $\bar{\partial}$-closed 1-form with coefficients in Hamiltonian vector fields. Assume that $H^2(\mathcal{O}_M) = 0$, that is, the holomorphic cohomology of the structure sheaf vanish. Then (2.2) has a solution.*

Proof. Let us write (2.2) in terms of the diagram (2.3), using the isomorphism $TM \cong \Lambda^{1,0}$. Using induction, we may assume that

$$\bar{\partial} \sum_{i+j=n} [\gamma_i, \gamma_j] = \sum_{i+j+k=n} [\gamma_i, [\gamma_j, \gamma_k]] + [[\gamma_i, \gamma_j,]\gamma_k],$$

which is equal zero by Jacoby identity. Therefore,

$$(2.4) \qquad\qquad \sum_{i+j=n} [\gamma_i, \gamma_j]$$

is $\bar{\partial}$-closed; to solve (2.2) we need to show that it is $\bar{\partial}$-exact, that is, to prove that it represents a zero class in the cohomology $H^2(\mathcal{H}am(M))$. The Hamiltonian vector fields are identified with ∂-closed $(1,0)$-forms. Poincare lemma gives an exact sequence of sheaves

$$(2.5) \qquad\qquad 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_M \xrightarrow{\partial} \mathcal{H}am(M) \longrightarrow 0,$$

Consider the piece

$$\longrightarrow H^2(\mathcal{O}_M) \longrightarrow H^2(\mathcal{H}am) \xrightarrow{h} H^3(M) \longrightarrow$$

of the corresponding long exact sequence. By Lemma 2.1, the expression (2.4) is ∂-exact; therefore, it represents zero in $H^3(M)$. By our assumptions,

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$H^2(\mathcal{O}_M) = 0$, and therefore, the map $h : H^2(\mathcal{H}am) \longrightarrow H^3(M)$ is an embedding. This proves that the sum (2.4) represents zero in $H^2(\mathcal{H}am(M))$, and there exists $\gamma_n \in \Lambda^{0,1}(\mathcal{H}am(M))$ such that $\bar{\partial}\gamma_n = \sum_{i+j=n}[\gamma_i, \gamma_j]$. We proved Theorem 2.2. $\qquad\square$

In Theorem 2.2 we identified the deformation space of $\langle M, \Omega \rangle$ with the cohomology of $\mathcal{H}am(M)$. These cohomology spaces are very easy to write down explicitly. Writing the long exact sequence corresponding to (2.5), we obtain

$$H^1(M) \longrightarrow H^1(\mathcal{O}_M) \longrightarrow H^1(\mathcal{H}am(M)) \longrightarrow H^2(M) \longrightarrow H^2(\mathcal{O}_M).$$

This gives an identification of the deformation space with $H^2(M)$ when $H^1(\mathcal{O}_M) = H^2(\mathcal{O}_M) = 0$, and, more generally, allows one to express the holomorphic symplectic deformations through $H^i(M)$, $H^i(\mathcal{O}_M)$ $(i = 1, 2)$.

In the remaining part of the paper, we combine the intuition of the *DG*-algebra approach with the hard science of Grothendieck's local deformations theory, and obtain essentially the same results in a much more rigorous setting.

3   Statement of the results.

3.1   Admissible manifolds. To begin with, we will describe the restrictions that we need to impose on the given holomorphically symplectic manifold X.

Let X be a smooth algebraic manifold over $\mathbb{C}$. By a *deformation* of X over a pointed scheme $\langle S, o \in S \rangle$ we will understand a scheme $\widetilde{X}/S$ smooth over S and equipped with an isomorphism $o \times_S \widetilde{X} \cong X$. We will only consider deformations over spectra $S = \operatorname{Spec} A$ of local Artin $\mathbb{C}$-algebra A, so that the fixed point is given by the maximal ideal $\mathfrak{m} \subset A$. Unless otherwise mentioned, all deformations will be assumed to be of this type.

Recall that the de Rham complex $\Omega^\bullet(X)$ is equipped with the Hodge, a.k.a. stupid filtration $F^\bullet\Omega^\bullet(X)$ given by

$$F^i\Omega^j(X) = \begin{cases} \Omega^j(X), & j \geq i, \\ 0, & \text{otherwise.} \end{cases}$$

This filtration is also defined, and by the same formula, for the relative de Rham complex $\Omega^\bullet(\widetilde{X}/S)$ of an arbitrary deformation $\widetilde{X}/S$.

For symplectic deformations, it is the first term $F^1\Omega^\bullet(X)$ of the Hodge filtration that plays the crucial role.

The complex $F^1\Omega^\bullet(X)$ can be included in an obvious exact triangle

$$(3.1) \qquad F^1\Omega^\bullet(X) \longrightarrow \Omega^\bullet(X) \longrightarrow \mathcal{O}_X \longrightarrow$$

where $\mathcal{O}_X$ is the structure sheaf of the manifold X. This triangle induces an exact triangle on cohomology.

Definition 3.1. A smooth algebraic manifold X over $\mathbb{C}$ is called *admissible* if for any deformation $\pi : \widetilde{X} \to S$ the relative cohomology sheaf

$$R^2\pi_*F^1\Omega^\bullet(\widetilde{X}/S)$$

is a flat sheaf on S and the canonical map

$$R^2\pi_*F^1\Omega^\bullet(\widetilde{X}/S) \to R^2\pi_*\Omega^\bullet(\widetilde{X}/S)$$

is injective.

This definition is pretty technical, because it is given in the most general form. For all practical applications that we see at the moment, it suffices to assume the stronger condition on X provided by the following easy lemma.

Lemma 3.2. *Let X be a smooth complex algebraic manifold. If for all $p \geq 1$ the canonical map*

$$H^p(X, \mathbb{C}) \to H^p(X, \mathcal{O}_X)$$

is surjective, then the manifold X is admissible.

Proof. Let $\pi : \widetilde{X} \to S$ be an arbitrary deformation. The existence of the Gauss-Manin connection implies that

$$R^p\pi_*\Omega^\bullet(\widetilde{X}/S) \cong H^p(X, \mathbb{C}) \otimes \mathcal{O}_S$$

for every $p \geq 0$. We will prove that

(A) for every $p \geq 2$, the canonical map $R^p\pi_*\Omega^\bullet(\widetilde{X}/S) \to R^p\pi_*\mathcal{O}(\widetilde{X})$ is surjective and the sheaf $R^p\pi_*\mathcal{O}(\widetilde{X})$ is flat on S.

Use downward induction on p, starting with an arbitrary $p > 2\dim X$. Assume (A) proved for all $p > k$. Denote by $i : o \to S$ the embedding of the

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base point. Consider the restriction of the cohomology exact triangle induced by (3.1) to the base point $o \in S$. Then base change and the Nakayma Lemma immediately imply that the map

$$R^k \pi_* \Omega^\bullet(\widetilde{X}/S) \to R^k \pi_* \mathcal{O}(\widetilde{X})$$

is surjective. This together with the inductive assumption implies that

(B) the canonical map $R^p \pi_* F^1 \Omega^\bullet(\widetilde{X}/S) \to R^p \pi_* \Omega^\bullet(\widetilde{X}/S)$ is injective and the sheaf $R^p \pi_* F^1 \Omega^\bullet(\widetilde{X}/S)$ if flat over S for every $p \geq k+1$.

Therefore if the sheaf $R^k \pi_* \mathcal{O}(\widetilde{X})$ is not flat, then the coboundary map

$$L^1 i^* R^k \pi_* \mathcal{O}(\widetilde{X})) \to i^* R^k \pi_* F^1 \Omega^\bullet(\widetilde{X}/S)$$

is not zero, which contradicts the assumption. Carrying the induction half-step further, we derive that the map

$$R^1 \pi_* \Omega^\bullet(\widetilde{X}/S) \to R^1 \pi_* \mathcal{O}(\widetilde{X})$$

is surjective. Therefore (B) also holds for $p = 2$, which proves the lemma. $\square$

This lemma shows that a manifold X is admissible in the following cases:

(i) X is compact (Hodge theory).

(ii) X is affine ($H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$).

(iii) More generally, X admits a proper generically one-to-one map $\pi : X \to Y$ into an affine variety Y with rational singularities (again $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$).

(iv) X admits a proper generically one-to-one map $\pi : X \to Y$ into an affine variety Y and has trivial canonical bundle K_X ($H^i(X, \mathcal{O}_X) = H^i(X, K_X) = 0$ for $i > 0$. [1]

If X is an admissible manifold, one can choose a splitting $H^2(X, \mathbb{C}) \to H^2(X, F^1 \Omega^\bullet(X))$ of the canonical surjection $H^2(X, F^1 \Omega^\bullet(X)) \to H^2(X, \mathbb{C})$. Together with the Gauss-Manin connection, this splitting defines an isomorphism

$$R^2 \pi_* F^1 \Omega^\bullet(\widetilde{X}/S) \cong H^2(X, F^1 \Omega^\bullet(X)) \otimes \mathcal{O}_S$$

[1]This follows immediately from the Grauert-Riemenschneider Vanishing Theorem, [GPR].

for every deformation $\pi : \widetilde{X} \to S$. We will always assume given such a splitting, keeping in mind that this introduces into our constructions an element of choice. Note that this choice does *not* appear in the case of the affine X, since in this case for every deformation we have $R^2\pi_*\mathcal{O}(\widetilde{X}) = 0$ and $R^2\pi_*F^1\Omega^{\bullet}(\widetilde{X}/S) \cong R^2\pi_*\Omega^{\bullet}(\widetilde{X}/S)$.

3.2 Symplectic deformations and the period map. Let X be an admissible manifold. Assume from now on that the manifold X is equipped with a non-degenerate closed 2-form $\Omega \in \Omega^2(X)$.

Definition 3.3. A *symplectic deformation* $\widetilde{X}/S$ of the symplectic manifold X over a base S is a usual deformation $\pi : \widetilde{X} \to S$ equipped with a closed relative 2-form $\Omega \in \Omega^2(\widetilde{X}/S)$ which becomes the given 2-form under the isomorphism $o \times_S \widetilde{X} \cong X$.

For every local Artin scheme $S = \operatorname{Spec} A$, we will denote by $\mathsf{Def}(X, S)$ or simply by $\mathsf{Def}(S)$ the set of isomorphism classes of symplectic deformations $\widetilde{X}/S$ of X over S.

Choose once and for all a splitting $H^2(X, \mathbb{C}) \to H^2(X, F^1\Omega^{\bullet}(X))$, so that for every deformation $\pi : \widetilde{X} \to S$ we have an isomorphism

$$R^2 F^1\Omega^{\bullet}(\widetilde{X}/S) \cong H^2(X, F^1\Omega^{\bullet}(X)) \otimes \mathcal{O}_S.$$

If the deformation $\widetilde{X}/S$ is symplectic, then the relative 2-form $\Omega \in \Omega^2(\widetilde{X}/S)$ defines a canonical cohomology class

$$[\Omega] \in H^2(X, F^1\Omega^{\bullet}(X)) \otimes \mathcal{O}_S.$$

This class gives a scheme map

$$\mathsf{Per}(\widetilde{X}) : S \to Tot(H^2(X, F^1\Omega^{\bullet}(X)),$$

where $Tot(H^2(X, F^1\Omega^{\bullet}(X))$ denotes the total space of $H^2(X, F^1\Omega^{\bullet}(X))$ considered as a scheme. Further on, we shall simplify notation by omitting "Tot".

Definition 3.4. The *period domain* of the admissible symplectic manifold X is the completion of the vector space $H^2(X, F^1\Omega^{\bullet}(X))$ near the point $[\Omega] \in H^2(X, F^1\Omega^{\bullet}(X))$ corresponding to the symplectic form $\Omega \in \Omega^2(X)$.
   The map $\mathsf{Per}(\widetilde{X})$ is called *the period map* of the deformation $\widetilde{X}/S$.

Note that the period map by definition maps S into the period domain $\mathsf{Per} \subset H^2(X, F^1\Omega^{\bullet}(X))$.

Remark 3.5. This definition of the period domain is essentially cheating: in Bogomolov's theory the period domain is not a formal scheme, but a globally (and non-trivially) defined quadric in the projectivization of the vector space $H^2(X, \mathbb{C})$. However, since we work only with infinitesemal deformation, Definition 3.4 is sufficient for our purposes.

For any local Artin scheme S, taking the period map defines a map

$$\mathsf{Per} : \mathsf{Def}(S) \to \mathsf{Per}(S)$$

from the set of deformation classes over S to the set of S-points of the formal scheme Per. This map (which we will, by abuse of the language, also call the period map) is functorial in S (where $\mathsf{Def}(S)$ is considered as a functor by taking pullbacks). We can now state our main result.

Theorem 3.6. *Let X be an admissible manifold equipped with a symplectic 2-form $\Omega \in \Omega^2(X)$. Then for any local Artin scheme S, the period map*

$$\mathsf{Per} : \mathsf{Def}(S) \to \mathsf{Per}(S)$$

induces a set bijection from the set of isomorphism classes of symplectic deformations $\widetilde{X}/S$ to the set of S-points of the period domain Per.

In particular, there exists a (formal) symplectic deformation $\mathfrak{X}/\mathsf{Per}$ such that any deformation $\widetilde{X}/S$ is isomorphic to the pullback of $\mathfrak{X}$ by means of the period map $\mathsf{Per}(\widetilde{X}) : S \to \mathsf{Per}$.

Remark 3.7. *Let M satisfy $H^i(M, \mathcal{O}_N) = 0$ for $i \geq 1$. Then Theorem 3.6 holds. Moreover, the period map $\mathsf{Per} : \mathsf{Def}(S) \to H^2(M)$ is locally an isomorphism.*

Remark 3.8. *For M affine, we obviously have $H^i(M, \mathcal{O}_M) = 0$ for $i \geq 1$. $H^1(\mathcal{O}_N) = H^2(\mathcal{O}_M) = 0$ obviously holds. In this case, Theorem 3.6 can be proved by a standard inductive argument (see Introduction).*

4 Elementary extensions and the Kodaira-Spencer class.

4.1 Elementary extensions. To prove Theorem 3.6, we will use induction on the length of the local Artin algebra $A = \mathcal{O}(S)$. To set up the induction, we introduce the following.

Definition 4.1. Let $S_0 \subset S$ be a closed embedding of local Artin schemes, and let $\widetilde{X}_0/S_0$ be a symplectic deformation of a holomorphically symplectic manifold X over S_0.

Then by $\mathsf{Def}(\widetilde{X}_0, S)$ we will denote the set of isomorphism classes of symplectic deformations $\widetilde{X}/S$ equipped with a symplectic isomorphism $\widetilde{X} \otimes_S S_0 \cong \widetilde{X}_0$.

This is consistent with our earlier notation. Indeed, by Definition 3.3, every symplectic deformation is canonically trivialized over the base point $o \subset S$. Therefore $\mathsf{Def}(X, S)$ in the sense of Definition 4.1 is still the set of isomorphism classes of all symplectic deformations of the manifold X.

The corresponding notion on the "period" side is the following.

Definition 4.2. Let $p_0 : S_0 \to \mathsf{Per}$ be a map from the closed subscheme $S_0 \subset S$ to the period domain Per. Then by $\mathsf{Per}(p_0, S)$ we will denote the set of all maps $f : S \to \mathsf{Per}$ such that $F|_{S_0} = p_0$.

It is customary in deformation theory to prove theorems step-by-step, starting with the case of square-zero extensions. However, we will need a slightly smaller class of extensions $S_0 \subset S$. Namely, let A be a local Artin algebra, and let $I \subset A$ be a square-zero ideal, so that $I^2 = 0$. Then we have the usual exact sequence of the modules of Kähler differentials over $\mathbb{C}$

$$(4.1) \qquad I \longrightarrow \Omega^1(A)/I \longrightarrow \Omega^1(A/I) \longrightarrow 0.$$

Definition 4.3. The extension $\mathrm{Spec}\, A/I \subset \mathrm{Spec}\, A$ will be called *elementary* if the sequence (4.1) is also exact on the left.

The following lemma immediately implies that every local Artin scheme S admits a filtration $o \subset S_0 \subset \ldots \subset S_k = S$ such that all extensions $S_i \subset S_{i+1}$ are elementary.

Lemma 4.4. *Let A be a local Artin algebra with the maximal ideal $\mathfrak{m}$. Assume that $\mathfrak{m}^{p+1} = 0$, while $\mathfrak{m}^p \neq 0$. Then the extension $\mathrm{Spec}\, A/\mathfrak{m}^p \subset \mathrm{Spec}\, A$ is elementary.*

Proof. We have to prove that the canonical map $\mathfrak{m}^p \to \Omega^1(A)/\mathfrak{m}^p$ is injective. It suffices to prove this for A replaced with its associated graded quotient with respect to the $\mathfrak{m}$-adic filtration. Thus we can assume that A and $\Omega^1(A)$ are graded. The map $\mathfrak{m}^p \to \Omega^1(A)/\mathfrak{m}^p$ is the composition of the de Rham

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differential $d : \mathfrak{m}^p \to \Omega^1(A)$ and the projection $\Omega^1(A) \to \Omega^1(A)/\mathfrak{m}^p$. Since d is obviously injective, it suffices to prove that $d(\mathfrak{m}^p) \cap \mathfrak{m}^p \cdot \Omega^1(A) = 0$. But this is trivial: $d(\mathfrak{m}^p)$ has degree p with respect to the grading on $\Omega^1(A)$, while $\mathfrak{m}^p \cdot \Omega^1(A)$ is of degree $(p+1)$. $\qquad\square$

This lemma reduces Theorem 3.6 to the following claim.

Proposition 4.5. *Let X be an admissible symplectic manifold. Let $S_0 \subset S$ be an elementary extension of local Artin schemes, and let $\widetilde{X}_0/S_0$ be an arbitrary symplectic deformation of the manifold X. Denote by $p_0 : S_0 \to$ Per the period map of the deformation $\widetilde{X}_0/S_0$.*

Then the period map

$$\mathsf{Per} : \mathsf{Def}(\widetilde{X}_0, S) \to \mathsf{Per}(p_0, S)$$

is as isomorphism.

4.2 The Kodaira-Spencer class. In order to start the proof of Proposition 4.5, we will need a convenient description of the set $\mathsf{Per}(p_0, S)$. To give such a description, we will consider not the period map itself, but its differential.

Definition 4.6. Let $\widetilde{X}/S$ be a symplectic deformation of a symplectic manifold X. Then the class

$$\theta = \nabla\Omega \in H^2(\widetilde{X}, F^1\Omega^{\bullet}(\widetilde{X}/S)) \otimes_{\mathcal{O}(S)} \Omega^1(S)$$

obtained by application of the Gauss-Manin connection ∇ to the relative 2-form Ω is called the *Kodaira-Spencer class* of the deformation $\widetilde{X}/S$.

Note that the symplectic form Ω is a cohomology class of the complex $F^2\Omega^{\bullet}(\widetilde{X}/S)$. Since the Gauss-Manin connection decreases the Hodge filtration at most by 1, the Kodaira-Spencer class θ is well-defined for an arbitrary symplectic manifold X.

When the symplectic manifold X is admissible, the Kodaira-Spencer class essentially coincides with the codifferential of the period map $\mathsf{Per}(\widetilde{X}) : S \to \mathsf{Per}$. More precisely, the codifferential

(4.2) $$\delta\,\mathsf{Per}(\widetilde{X}) : \mathsf{Per}^* \Omega^1(\mathsf{Per}) \to \Omega^1(S)$$

of the period map is given by

$$\delta\,\mathsf{Per}(\widetilde{X})(\alpha) = \langle \alpha, \theta \rangle,$$

where the bundle $\mathsf{Per}^* \, \Omega^1(\mathsf{Per})$ is identified with the trivial bundle

$$\left(H^2(X, F^1\Omega^{\bullet}(X))\right)^* \otimes \mathcal{O}_S,$$

α is an arbitrary section of this trivial bundle, and $\langle \bullet, \bullet \rangle$ means the pairing on the first factor in

$$H^2(\widetilde{X}, F^1\Omega^{\bullet}(\widetilde{X}/S)) \otimes_{\mathcal{O}(S)} \Omega^1(S).$$

Let now $i : S_0 \hookrightarrow S$ be an elementary extension, let $\widetilde{X}_0/S_0$ be a symplectic deformation of a symplectic manifold X, and let

$$\theta_0 \in H^2(\widetilde{X}_0, F^1\Omega^{\bullet}(\widetilde{X}_0/S_0)) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0)$$

be its Kodaira-Spencer class. Denote by $\eta : i^*\Omega^1(S) \to \Omega^1(S_0)$ the canonical surjection of the modules of differentials.

Definition 4.7. By $KS(\theta_0, S)$ we will denote that set of all cohomology classes

$$\theta \in H^2(\widetilde{X}_0, F^1\Omega^{\bullet}(\widetilde{X}_0/S_0)) \otimes_{\mathcal{O}(S_0)} i^*\Omega^1(S)$$

such that

$$\eta(\theta) = \theta_0 \in H^2(\widetilde{X}_0, F^1\Omega^{\bullet}(\widetilde{X}_0/S_0)) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0).$$

Assume that X is admissible, so that for every symplectic deformation we have the period map, and denote by $p_0 : S_0 \to \mathsf{Per}$ the period map of the deformation $\widetilde{X}_0/S_0$. It is the set $KS(\theta_0, S)$ which we will use as a model for the set $\mathsf{Per}(p_0, S)$. To do this, notice that every element $p \in \mathsf{Per}(p_0, S)$ defines an element $\theta(p) \in KS(\theta_0, S)$ by the formula (4.2). This correspondence is in fact one-to-one.

Lemma 4.8. *The correspondence $p \mapsto \theta(p)$ is a bijection between the set $\mathsf{Per}(p_0, S)$ and the set $KS(\theta_0, S)$.*

Proof. Indeed, both sets are torsors over the group

$$H^2(\widetilde{X}_0, F^1\Omega^{\bullet}(\widetilde{X}_0/S_0)) \otimes_{\mathcal{O}(S_0)} I,$$

where $I \subset \mathcal{O}(S)$ is the kernel of the map $\mathcal{O}(S) \to \mathcal{O}(S_0)$. For the group $\mathsf{Per}(p_0, S)$, this is obvious from the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}(S) \longrightarrow \mathcal{O}(S_0) \longrightarrow 0,$$

while for the set $KS(\theta_0, S)$ this follows from the exact sequence (4.1) of the differentials (exact on the left since the extension $S_0 \subset S$ is elementary). $\square$

The advantages of the set $KS(\theta_0, S)$ over the set $\mathsf{Per}(p_0, S)$ are twofold. Firstly, Definition 4.7 can be refined so that it takes account of automorphisms of deformation of X. This will be explained in the next section. Secondly, and this we will state now, Definition 4.7 is essentially local on X. Namely, we have the following obvious fact.

Lemma 4.9. *The set $KS(\theta_0, S)$ can be equivalently defined as the set of all maps*
$$\theta \in \mathrm{Hom}(\mathcal{O}(S_0)[-2], F^1\Omega^\bullet(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} i^*\Omega^1(S))$$
in the derived category of sheaves of $\mathcal{O}(S_0)$-modules on X, such that $\eta(\theta) = \theta_0$. Here $\mathcal{O}(S_0)$ is considered as the constant sheaf. $\square$

Because of this, we can use Definition 4.7 to restate Proposition 4.5 without the admissibility condition on the manifold X.

Proposition 4.10. *Let $S_0 \subset S$ be an elementary extension of local Artin algebras. Let X be a symplectic manifold, let $\widetilde{X}_0/S_0$ be a symplectic deformation, and let*
$$\theta_0 \in \mathrm{Hom}(\mathcal{O}(S_0)[-2], F^1\Omega^\bullet(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0))$$
be its Kodaira-Spencer class.
Then associating to a symplectic deformation $\widetilde{X}/S$ its Kodaira-Spencer class defines a bijection
$$\mathsf{Per}(\widetilde{X}_0, S) : \mathsf{Def}(\widetilde{X}_0, S) \cong KS(\theta_0, S).$$

5 Localization, stacks and the proof of the main theorem.

5.1 Groupoids. We can now begin the proof of Proposition 4.10, hence also of Theorem 3.6.

Assume given a symplectic manifold X, an elementary extension $i : S_0 \hookrightarrow S$ and a symplectic deformation $\widetilde{X}_0/S_0$. Denote by
$$\theta_0 \in \mathrm{Hom}(\mathcal{O}(S_0)[-2], F^1\Omega^\bullet(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0))$$
the Kodaira-Spencer class of the deformation $\widetilde{X}_0/S_0$.

To begin with, we will refine the definitions of the sets $\mathsf{Def}(\widetilde{X}_0, S)$ and $KS(\theta_0, S)$ and of the map

$$\mathsf{Per}(\widetilde{X}_0, S) : \mathsf{Def}(\widetilde{X}_0, S) \to KS(\theta_0, S)$$

so as to take into account possible automorphisms of the deformations $\widetilde{X} \in \mathsf{Def}(\widetilde{X}_0, S)$. Such deformations naturally form a category. Moreover, this category is obviously a groupoid (i.e. all morphisms are invertible).

Definition 5.1. By $\mathcal{D}ef(\widetilde{X}_0, S)$ we will denote the groupoid of all symplectic deformations $\widetilde{X}/S$ equipped with an isomorphism $\widetilde{X} \times_S S_0 \cong \widetilde{X}_0$.

The set $\mathsf{Def}(\widetilde{X}_0, S)$ is by definition the set of objects in the groupoid $\mathcal{D}ef(\widetilde{X}_0, S)$.

To construct a groupoid version of the set $KS(\theta_0, S)$, note that the elements

$$\theta \in \mathrm{Hom}(\mathcal{O}(S_0)[-2], F^1\Omega^{\bullet}(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0))$$

naturally classify exact sequences

$$0 \longrightarrow F^1\Omega^{\bullet}(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} i^*\Omega^1(S) \longrightarrow \bullet \longrightarrow \mathcal{O}(S_0) \longrightarrow 0$$

in the (abelian) category of complexes of sheaves of $\mathcal{O}(S_0)$-modules on X. Such an element satisfies $\eta(\theta) = \theta_0$ if and only if there exists a commutative diagram

(5.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F^1\Omega^{\bullet}(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} i^*\Omega^1(S) & \longrightarrow & \bullet & \longrightarrow & \mathcal{O}(S_0) & \longrightarrow & 0 \\
& & \Big\downarrow {\mathsf{id}\,\otimes\eta} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & F^1\Omega^{\bullet}(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0) & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{O}(S_0) & \longrightarrow & 0
\end{array}
,
$$

whose the bottom row is the exact sequence corresponding to the given class

$$\theta_0 \in \mathrm{Hom}(\mathcal{O}(S_0)[-2], F^1\Omega^{\bullet}(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0)),$$

and $\mathcal{L}$ is the associated extension.

This motivates the following.

Definition 5.2. By $\mathcal{KS}(\theta_0, S)$ we will denote the groupoid whose objects are commutative diagrams of the type (5.1), and whose morphisms are maps between these commutative diagrams identical everywhere except for $\bullet$.

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Again, by definition the set $KS(\theta_0, S)$ is the set of isomorphism classes of objects in the groupoid $\mathcal{KS}(\theta_0, S)$. Moreover, the period map

$$\mathsf{Per}(\widetilde{X}_0, S) : \mathsf{Def}(\widetilde{X}_0, S) \cong KS(\theta_0, S)$$

lifts to a functor

(5.2) $$\mathcal{P}er(\widetilde{X}_0, S) : \mathcal{D}ef(\widetilde{X}_0, S) \cong \mathcal{KS}(\theta_0, S).$$

This is not immediately obvious, since an element in a first Ext-group defines a short exact sequence only up to a non-canonical isomorphism. However, in our case there is a canonical choice for a short exact sequence (5.1). Namely, for every deformation $\pi : X \to S$ we have a two-step filtration $\pi^*\Omega^1(S) \subset \Omega^1(X)$ on the sheaf of Kähler differentials $\Omega^1(X)$, with the quotient $\Omega^1(X/S)$. This filtration induces a filtration on the total de Rham complex $\Omega^\bullet(X)$. Taking only the top two quotients of this filtration, we obtain a canonical short exact sequence

(5.3)
$$0 \longrightarrow \pi^*\Omega^1(S) \otimes \Omega^\bullet(X/S) \longrightarrow \bullet \longrightarrow \Omega^{\bullet+1}(X/S) \longrightarrow 0$$

of complexes on X and the corresponding extension class

$$\eta \in \mathrm{Ext}^1(\Omega^{\bullet+1}(X/S), \pi^*\Omega^1(S) \otimes \Omega^\bullet(X/S)).$$

The class η essentially induces the Gauss-Manin connection: for every relative cohomology class

$$\alpha \in H^\bullet(X, \Omega^\bullet(X/S)) = \mathrm{Ext}^\bullet(\mathcal{O}(X), \Omega^\bullet(X/S)),$$

the class

$$\nabla(\alpha) \in H^\bullet(X, \Omega^\bullet(X/S)) \otimes \Omega^1(S)$$

is equal to $\rho \circ \alpha$. Moreover, the sequence (5.3) is compatible with the Hodge filtration. In particular, we have a canonical short exact sequence

(5.4)
$$0 \longrightarrow \pi^*\Omega^1(S) \otimes F^1\Omega^\bullet(X/S) \longrightarrow \bullet \longrightarrow F^2\Omega^\bullet(X/S) \longrightarrow 0.$$

Now, by definition of the Kodaira-Spencer class θ we have

$$\theta = \nabla(\Omega) = \eta \circ \Omega.$$

Thus the short exact sequence (5.1) corresponding to the deformation X/S is obtained by composing the canonical short exact sequence (5.4) with the map $\mathcal{O}(X)[2] \to F^2\Omega^\bullet(X/S)$ given by Ω.

The reader can easily see that this construction is completely functorial, so that we indeed have a functor (5.2). We will call it the *period functor.*

Our third and final reformulation of Theorem 3.6 is the following.

Proposition 5.3. *Under the assumption of Proposition 4.10, the period functor*
$$\mathcal{P}er(\widetilde{X}_0, S) : \mathcal{D}ef(\widetilde{X}_0, S) \cong \mathcal{KS}(\theta_0, S)$$
is an equivalence of categories.

This proposition implies Proposition 4.10, hence also Proposition 4.5 and Theorem 3.6.

5.2   Reduction to the affine case. The following lemma explains why we introduced the groupoids.

Lemma 5.4. *Assume that Proposition 5.3 holds for affine symplectic manifolds X. Then it holds for an arbitrary symplectic manifold X.*

Proof. For every open subset $U \subset X$, the deformation $\widetilde{X}_0/S_0$ induces a symplectic deformation $\widetilde{U}_0/S_0$ of the manifold U. The collection of groupoids $\mathcal{D}ef(\widetilde{U}_0, S)$ is a stack on X in Zariski topology.

Moreover, for every open $U \subset X$ any diagram of the type (5.1) induces by restriction a diagram of the same type for the manifold U, and the collection of groupoids $\mathcal{KS}(\theta_0|_U, S)$ is also a stack on X in Zariski topology (it is to insure this that we have chosen to work with exact sequences of complexes on X instead of using the derived category).

The period functors $\mathcal{P}er(\widetilde{U}_0, S) : \mathcal{D}ef(\widetilde{U}_0, S) \to \mathcal{KS}(\theta_0|_U, S)$ is a functor between these stack. Since by assumption it is an equivalence for affine $U \subset X$, it is an equivalence for an arbitrary $U \subset X$ – in particular, for X itself. $\square$

5.3   The affine case. It remains to prove Proposition 5.3 in the case of an affine symplectic manifold X. This is essentially done in the Introduction. Here we state the same argument in a more refined language.

Assume given an affine symplectic manifold X, an elementary extension $S_0 \subset S$, and a symplectic deformation $\widetilde{X}_0/S_0$ with Kodaira-Spencer class θ_0.

Since X is affine and smooth, every deformation $\widetilde{X}/S$ is trivial as an algebraic manifold. Thus we can choose an isomorphism $\widetilde{X}_0 \cong X \times S_0$, and every object $\widetilde{X} \in \mathsf{Ob}\,\mathcal{D}ef(\widetilde{X}_0, S)$ is isomorphic as an algebraic manifold to $X \times S$. We introduce the following notation.

Definition 5.5. If a group G acts on a set N, then the *quotient groupoid* N/G is the groupoid whose objects are elements $n \in N$ and whose morphisms are given by

$$\mathrm{Hom}(n_1, n_2) = \{g \in G \mid g \cdot n_1 = n_2\}, \qquad n_1, n_2 \in N.$$

Then the groupoid $\mathcal{D}ef(\widetilde{X}_0, S)$ is equivalent to the quotient groupoid

$$\mathsf{Sympl}\,/\,\mathsf{Aut},$$

where Sympl is the set of all relative closed 2-forms $\Omega \in \Omega^2(X \times S/S)$ whose restriction to $\widetilde{X}_0 = X \times S_0 \subset X \times S$ coincides with the given symplectic form $\Omega_0 \in \Omega^2(\widetilde{X}_0/S_0)$, and Aut is the group of automorphisms of the algebraic manifold $X \times S$ which commute with the projection $X \times S \to S$ and which are identical on $\widetilde{X}_0 \subset X \times S$.

Every form $\Omega \in \mathsf{Sympl}$ can be represented as

$$\Omega = \Omega_0 + \beta,$$

where $\beta \in \Omega^2(X) \otimes I$ is a closed 2-form on X with values in the ideal $I = \mathrm{Ker}(\mathcal{O}(S) \to \mathcal{O}(S_0))$. Therefore we have a canonical identification $\mathsf{Sympl} = \Omega_{cl}^2(X) \otimes I$ of Sympl with the space $\Omega_{cl}^2 X) \otimes I$ of closed I-valued 2-forms on X. Moreover, every automorphism $g \in \mathsf{Aut}$ is an automorphism of the function ring $\mathcal{O}(X \times S)$ of the form

$$g = \mathsf{id} + \xi,$$

where $\xi \in \mathcal{T}(X) \otimes I$ is an I-valued vector field on X. Since $I \subset \mathcal{O}(S)$ is a square-zero ideal, the group Aut is commutative and isomorphic to the abelian group $\mathcal{T}(X) \otimes I$ – that is, we have

$$(\mathsf{id} + \xi_1) \cdot (\mathsf{id} + \xi_2) = \mathsf{id} + \xi_1 + \xi_2$$

for every $\xi_1, \xi_2 \in \mathcal{T}(X) \otimes I$. Finally, by Cartan homotopy formula, the action of Aut on Sympl is given by

$$(\mathsf{id} + \xi) \cdot (\Omega_0 + \beta) = \Omega_0 + \beta + d(\Omega_0 \lrcorner \xi).$$

where d is the de Rham differential (all the other terms vanish since $I^2 = 0$). To sum up, we have

$$\mathcal{D}ef(\widetilde{X}_0, S) \cong \Omega^2_{cl}(X) \otimes I / \mathcal{T}(X) \otimes I,$$

and the action is given by

$$\xi \cdot \beta = \beta + \Omega_0 \lrcorner \xi.$$

We will now describe the right-hand side of the hypothetical equivalence (5.2) – that is, the groupoid $\mathcal{KS}(\theta_0, S)$ – in a similar way. To do this, notice that since X is affine, we can replace sheaves on X with the modules of their global sections. Moreover, the trivialization $\widetilde{X}_0 \cong X \times S_0$ provides identifications

$$F^1\Omega^\bullet(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} i^*\Omega^1(S) \cong F^1\Omega^\bullet(X) \otimes_{\mathbb{C}} \Omega^1(S)/I,$$
$$F^1\Omega^\bullet(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S_0) \cong F^1\Omega^\bullet(X) \otimes_{\mathbb{C}} \Omega^1(S_0),$$

of the complexes in the left column of a commutative diagram of type (5.1).

In every commutative diagram of complexes of this type, both rows split as exact sequences of graded vector spaces. The only possibly non-trivial extension data are contained in the differential of the complex $\bullet$. Denote the set of possible differentials by Diff. Analogously, every map between two diagrams of the type (5.1) must be an automorphism of $\bullet$ which is upper-triangular with respect to the splitting

$$\bullet = \left(F^1\Omega^\bullet(\widetilde{X}_0/S_0) \otimes_{\mathcal{O}(S_0)} \Omega^1(S)/I \right) \oplus \mathcal{O}(S_0).$$

If we denote the group of all such automorphisms by Tr, then the groupoid $\mathcal{KS}(\theta_0, S)$ is equivalent to the quotient groupoid Diff / Tr.

To identify the set Diff, note that every possible differential $\partial \in$ Diff, being a map of $\mathcal{O}(S_0)$-modules, is completely determined by its value

$$\partial(1) \in \Omega^2(X) \otimes \Omega^1(S)/I$$

on the unity $1 \in \mathcal{O}(S_0)$. Since the differential in the complex $\mathcal{L}$ from the bottom row of (5.1) is fixed, every two such differentials $\partial_1, \partial_2 \subset$ Diff must differ by a 2-form

$$\beta = \partial_1(1) - \partial_2(1) \in \Omega^2(X) \otimes I \subset \Omega^2(X) \otimes \Omega^1(S)/I.$$

Moreover, since every differential $\partial \subset$ Diff must satisfy $\partial^2 = 0$, the difference $\beta = \partial_1(1) - \partial_2(1)$ must be a closed I-valued 2-form. Thus the set Diff is a torsor over the abelian group $\Omega^2_{cl}(X) \otimes I$.

Analogously, every triangular map $g \in \mathsf{Tr}$ is of the form

$$g = \begin{pmatrix} \mathsf{id} & a \\ 0 & \mathsf{id} \end{pmatrix}$$

for some

$$a \in \mathrm{Hom}(\mathcal{O}(S_0), \Omega^1(X) \otimes I) \subset \mathrm{Hom}(\mathcal{O}(S_0), \Omega^1(X) \otimes \Omega^1(S)/I),$$

and composition of maps g_1, g_2 simply adds the associated elements a_1, a_2. Since a must be a map of $\mathcal{O}(S_0)$-modules, it is completely determined by

$$\alpha = a(1) \in \Omega^1(X) \otimes I.$$

Therefore Tr is the abelian group $\Omega^1(X) \otimes I$. Under these identifications, the action of Tr on Diff is given by

$$\alpha \cdot \beta = \beta + d\alpha,$$

where d is the de Rham differential.

Having done these identifications, we notice that the period functor $\mathcal{P}er : \mathcal{D}ef(\widetilde{X}_0, S) \to \mathcal{KS}(\theta_0, S)$ is defined on the level of quotient groupoids by maps

$$\mathsf{Sympl} \to \mathsf{Diff}$$
$$\mathsf{Aut} \to \mathsf{Tr}$$

The first map is tautologically an isomorphism $\Omega^2_{cl}(X) \otimes I \cong \Omega^2_{cl}(X) \otimes I$. The second is the map

$$\mathcal{T}(X) \otimes I \to \Omega^1(X) \otimes I$$

given by $\xi \mapsto \Omega_0 \lrcorner \xi$. It is an isomorphism because the symplectic 2-form $\Omega_0 \in \Omega^2(\widetilde{X}_0/S_0)$ is by assumption non-degenerate.

This finishes the proof of Proposition 5.3, hence, by a long chains of reductons, also proves Theorem 3.6. $\qquad\square$

6 Postface

To finish the paper, we would to give a few comments (perhaps a bit vague) as to how the theory of symplectic deformations is related to the usual

deformation theory, and what is the relation of this paper to the known results.

The period map is a purely symplectic phenomenon – it has no analogs in the usual deformation theory (although there might be a useful version for the deformation theory of Calabi-Yau manifolds). On the other hand, its differential, which we called the Kodaira-Spencer class, is a fairly general thing. Namely, recall that for every smooth family $\pi : X \to S$, say over an affine base S, there exist a canonical class

$$\theta \in \mathrm{Ext}^1(\Omega^1(X/S), \pi^*\Omega^1(S)),$$

– the extension class given by the exact sequence

$$0 \longrightarrow \pi^*\Omega^1(S) \longrightarrow \Omega^1(X) \longrightarrow \Omega^1(X/S) \longrightarrow 0$$

of differentials for the map $\pi : X \to S$. Since X/S is smooth, the sheaf $\Omega^1(X/S)$ is flat, and this class can be reinterpreted as a class in the first cohomology group

$$H^1(X, \mathcal{T}(X/S) \otimes \pi^*\Omega^1(S)) \cong H^1(X, \mathcal{T}(X/S)) \otimes \Omega^1(S)$$

of the relative tangent bundle $\mathcal{T}(X/S)$. It is this class that is usually called the Kodaira-Spencer class of the deformation X/S.

If the deformation X/S is symplectic, then the symplectic form $\Omega \in \Omega^2(X/S)$ identifies the relative tangent bundle $\mathcal{T}(X/S)$ with the relative cotangent bundle $\Omega^1(X/S)$ by means of the correspondence $\xi \mapsto \Omega \lrcorner \xi$. This makes it possible to compare the usual Kodaira-Spencer class

$$\theta \in H^1(X, \mathcal{T}(X/S)) \otimes \Omega^1(S) \cong H^1(X, \Omega^1(X/S) \otimes \Omega^1(S))$$

with the "symplectic" Kodaira-Spencer class $\widetilde{\theta}$ introduced in Definition 4.6. These classes essentially coincide:

Lemma 6.1. *For every symplectic deformation X/S, the usual Kodaira-Spencer class*
$$\theta = \xi \lrcorner \Omega \in H^1(X, \Omega^1(X/S) \otimes \Omega^1(S))$$
is obtained from the symplectic Kodaira-Spencer class

$$\widetilde{\theta} \in H^2(X, F^1\Omega^\bullet(X/S)) \otimes \Omega^1(S)) = H^1(X, F^1\Omega^\bullet(X/S)[1]) \otimes \Omega^1(S))$$

by the tautological projection

$$F^1\Omega^\bullet(X/S)[1] \to \Omega^1(X/S).$$

Proof. Indeed, by definition of the Gauss-Manin connection ∇ and the usual Kodaira-Spencer class $\xi \in H^1(X, \mathcal{T}(X/S)) \otimes \Omega^1(S)$, for every smooth family X/S and every global relative k-form $\alpha \in \Omega^k(X/S)$ we have

$$\nabla(\alpha) = \alpha \lrcorner \xi.$$

Applying this to our symplectic deformation X/S and to the symplectic form Ω gives the result. $\qquad\square$

In fact, Definition 4.7 of the groupoid $\mathcal{KS}(\widetilde{X}_0, S)$ is also general and works in the usual deformation theory. Moreover, the corresponding versions of Proposition 4.10 and Proposition 5.3 are also true (and proofs are more or less the same). However, in the usual case we do not have the period map, and the groupoid $\mathcal{KS}(\widetilde{X}_0, S)$ is not easy to describe. In particular, it might be empty – that is, there might be a homological obstruction to the existence of a commutative diagram of type (5.1). This is a very well-known phenomenon. In the symplectic case, the existence of the period map ensures that (for admissible manifolds) there are no obstructions for deformation at any step.

When one tries to describe usual deformations by means of the associated Kodaira-Spencer class (instead of a period map which does not exist in general), one enters a closed loop: the class θ, which theoretically should uniquely define a deformation X/S, lies in the group $H^1(X, \mathcal{T}(X/S))$ which itself depends on the deformation. The main technical idea in our proof of Theorem 3.6 is to avoid it by going step-by-step through elementary extensions and using the exact sequence of differentials

$$0 \longrightarrow I \longrightarrow \Omega^1(A)/I \longrightarrow \Omega^1(A/I) \longrightarrow 0$$

for an elementary extension $\operatorname{Spec} A/I \subset \operatorname{Spec} A$. This allows one to describe extension of a deformation $X_0/\operatorname{Spec}(A/I)$ to a deformation $X/\operatorname{Spec} A$ in terms of the lifting of the Kodaira-Spencer class from $\Omega^1(A/I)$ to $\Omega^1(A)/I$ – and this works, because the module $\Omega^1(A)/I$ is already defined over A/I. This idea (at least for one-parameter elementary extensions $\operatorname{Spec} \mathbb{C}[t]/t^k \subset \operatorname{Spec} \mathbb{C}[t]/t^{k+1}$) is due entirely to Z. Ran [R]. We believe that it is this technique that he called *the T_1-lifting property.*

We would also like to notice that all the obstructions to symplectic deformations vanish essentially because for an admissible manifold X, the sheaf $R^2\pi_*(\widetilde{X}, F^1\Omega^\bullet(\widetilde{X}/S))$ is flat on S for every deformation $\pi : X \to S$. The same thing happens in the Ran's proof of the Tian-Todorov Lemma – that is, the lack of obstructions for deformations of a compact Calabi-Yau manifold is due to the flatness of some canonically defined sheaves. However,

Ran works with the usual deformations, and he (in the notation as above) needs the flatness of two sheaves: $\pi_* \Omega^n(\widetilde{X}/S)$ and $R^1\pi_* \Omega^{n-1}(\widetilde{X}/S)$ (here $n = \dim X$). This flatness is provided by Hodge theory. In the symplectic version of the proof, one would use instead the sheaves $\pi_* \Omega^2(\widetilde{X}/S)$ and $R^1\pi_* \Omega^1(\widetilde{X}/S)$.

From this point of view, the only new thing in our paper is the following observation: if one agrees to consider deformations that are *a priori* symplectic, then one can combine $\pi_* \Omega^2(\widetilde{X}/S)$ and $R^1\pi_* \Omega^1(\widetilde{X}/S)$ into $R^2\pi_* F^1\Omega^\bullet(\widetilde{X}/S)$ – and the latter sheaf is flat for a much wider class of manifolds X.

Finally, there's another, perhaps more conceptual explanation for the exceptional role played by the complex $F^1\Omega^\bullet(X)$ in the symplectic deformation theory. This explanation comes from the general deformation theory of algebras over an operad, sketched for example in [G1], [G2]. Symplectic manifolds can not be described by operad. However, a more general class of *Poisson manifolds* admits such a description. The general deformation theory for algebras over an operad works in a way completely parallel to the usual one, but the tangent bundle (more generally, the tangent complex) is replaced by its operadic version. For the Poisson operad and a symplectic manifold X, the Poisson tangent complex is precisely $F^1\Omega^\bullet(X)$ (independently of the symplectic form). For a more general Poisson structure, this is the complex $\Lambda^{\geq 1}\mathcal{T}(X)$ of polyvector fields of degree ≥ 1 on X, and the differential is given by the commutator with the Poisson bivector field $\Theta \in \Lambda^2\mathcal{T}(X)$.

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